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The signatures of finite dimensional representations of the de Sitter groups SO(4, 1) and SO(3, 2)

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Abstract. The signature of a finite-dimensional orthogonal representation of a simple Lie group is the difference between the number of positive and negative signs in the diagonal form of its symmetric bilinear invariant. We derive the expressions for the signatures of all finite-dimensional representations of the de Sitter groups SO(4, 1) and SO(3, 2) in two ways. One by means of character values of appropriate elements of adjoint order two on the representation and the other through generating functions.

1. Introduction

Let us first define the signature of a finite-dimensional representation of a simple Lie algebra over the complex or real field in a way which is more general than our immediate needs and more general than the traditional definition.

A signature of an irreducible representation Λ of a simple Lie algebra L over $\mathbb{C}$ is the character value on Λ of an (inner) automorphism of L whose adjoint order is two.

Note that under the definition such a Lie algebra has several signatures associated with it if its rank is greater than one. Indeed, for each conjugacy class of the automorphisms there is one signature. In the traditional way, the signatures are associated with non-compact real forms of simple Lie algebras. It is well known that there is a one-to-one correspondence between conjugacy classes of the automorphisms of order two of the algebras (over $\mathbb{C}$) and the real forms of the simple Lie algebra. We call each such automorphism the *defining automorphism* for the corresponding real form. In case a real form corresponds to a inner automorphism, i.e. the automorphism is induced by an element of the underlying group, then the definition above can be extended to the signature of a real form as follows.

The signature of an irreducible representation of a real form is the character value of the element defining the automorphism on the representation.

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Even that is not always the traditional signature. For two reasons. It is defined for all representations not only those with a symmetric bilinear invariant form (orthogonal representations), and it may differ from the usual definition by a phase factor. (There is an ambiguity of only a sign in the traditional definition of the signature.)

In order to make use of such a definition, one needs to be familiar with the general description of the automorphims of finite order and their action on representation spaces. Such a description is available [10] but it is not reproduced here in full generality since only a very special part of it is needed for our purposes in this paper. Nevertheless, we will draw on it when recalling these automorphisms. In addition, one needs to know which representations are orthogonal or symplectic or neither of the two. That information is readily available since the work of Dynkin [23], in particular see [24, p 5].

The question about signatures in some degree of generality (for all orthogonal representations of Lie algebras of a particular type) was first raised in [8]. Recently in a series of overlapping papers [17–21] another approach to the problem was taken and, in principle, signatures of orthogonal representations of the real forms (defined by inner automorphisms of order 2) of the classical Lie algebras as well as of the real forms of the exceptional simple Lie algebras were provided.

The definitions above offer, in our opinion, a general approach to the signature problem which is conceptually simpler and which can be quite practical, given the development of uniform and efficient methods of computing character values of elements of finite order on irreducible representations [10, 22].

The purpose of this paper is to provide the signatures in a limited series of cases, which have been studied in physics literature for many years in the context of various applications. We consider all finite-dimensional representations of the de Sitter groups SO(4, 1) and SO(3, 2).

We have two reasons for singling out these cases. First, the groups SO(4, 1) and SO(3, 2) appeared in physics as the isometry groups of the *de Sitter spaces*, which have a nonzero constant curvature, unlike the flat Minkowski spacetime [1–3]. The Lie algebras of the de Sitter groups can be 'deformed' into the Poincaré Lie algebra by a procedure called 'contraction' [4, 5, 15]. It was pointed out that these are the only simple Lie algebras which can be Wigner–Inönü contracted into the Poincaré algebra [6]. Moreover, it was shown in [5, 7] that all the 'kinematical groups' can be obtained by contractions of the de Sitter groups.

Our second reason is that the answer can be given in a very efficient way as a generating function for the signatures (i.e. for the corresponding character values) which are found already in literature [11–13]. In this form signatures in other cases would also be feasible to find. The structure of the generating function provides answers to uncommon related questions like the following: How many distinct signature values one finds among all the irreducible representations of the real form? What are these values? What are the representations with a given value of signature? These questions can be answered using the structure of the generating functions or by rearrangement of the terms in the power series expansion of the generating function [11].

2. Signatures and character values

Let $\sigma_\lambda\colon\ O(p,q)\to \mathrm{GL}(V)$ be the irreducible representation of $O(p,q)$ with highest weight λ, on a space V. Denote its dimension by d_λ and its weight system by Ω_λ. Note that in the case $p+q=5$ the representation σ_λ with highest weight $\lambda=(\lambda_1,\lambda_2)$ is orthogonal exactly when λ_1,λ_2 are non-negative integers and λ_1 is even. The diagonal form of the symmetric bilinear invariant B on V reads

$$(x, y) = x^{\mathrm{t}} B_\lambda y \qquad \text{with } B_\lambda = I_{p_\lambda} \oplus (-I_{q_\lambda}) \qquad \text{for all } x, y \in V. \tag{2.1}$$

Here I_n is the $n \times n$ identity matrix, and the superscript t denotes transposition.

The following simple observation is the central argument of our approach. Any automorphism of $O(p, q)$ induced by an element $X \in O(p, q)$ fixing the symmetric bilinear form B has to fulfil)

$$X^{\mathrm{t}} B_\lambda X = B_\lambda.$$

Note that B_λ itself corresponds to such an element in the group and is of order two, i.e. $B_\lambda^2 = I_{p+q}$.

The signature of the symmetric bilinear invariant of λ,

$$s_\lambda = p_\lambda - q_\lambda = \mathrm{Tr}(B_\lambda) \tag{2.2}$$

is the character value which the element represented by B_λ takes on the representation λ.

It is clear that (2.2) and

$$d_\lambda = p_\lambda + q_\lambda \tag{2.3}$$

imply

$$p_\lambda = \tfrac{1}{2}(d_\lambda + s_\lambda) \qquad \text{and} \qquad q_\lambda = \tfrac{1}{2}(d_\lambda - s_\lambda). \tag{2.4}$$

It is straightforward to get analogous quantities for direct sums and direct products of representations. For the direct sum of two irreducible representations $\lambda \oplus \mu$, we have

$$p_{\lambda\oplus\mu} = p_\lambda + p_\mu \qquad q_{\lambda\oplus\mu} = q_\lambda + q_\mu \qquad s_{\lambda\oplus\mu} = s_\lambda + s_\mu \qquad d_{\lambda\oplus\mu} = d_\lambda + d_\mu \tag{2.5}$$

and for the tensor product $\lambda \otimes \mu$,

$$p_{\lambda\otimes\mu} = p_\lambda p_\mu + q_\lambda q_\mu \qquad q_{\lambda\otimes\mu} = p_\lambda q_\mu + q_\lambda p_\mu \qquad s_{\lambda\otimes\mu} = s_\lambda s_\mu \qquad d_{\lambda\otimes\mu} = d_\lambda d_\mu. \tag{2.6}$$

It is a remarkable fact that the signatures of *non-compact* real forms (such as SO(4, 1) and SO(3, 2)) can be obtained from the values of characters on elements of finite order of their simply connected *compact* Lie group (here, Sp(4)). The complexification of its Lie algebra is $\mathfrak{sp}_4(\mathbb{C})$ of type C_2 with Cartan matrix

$$M = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}. \tag{2.7}$$

In terms of the simple roots α_1 and α_2, the fundamental weights are

$$\begin{aligned} \omega_1 &= (1, 0) = \alpha_1 + \tfrac{1}{2}\alpha_2 \\ \omega_2 &= (0, 1) = \alpha_1 + \alpha_2. \end{aligned} \tag{2.8}$$

The highest weight $\Lambda = \lambda_1\omega_1 + \lambda_2\omega_2$ which we write as $\Lambda = (\lambda_1, \lambda_2)$ corresponding to the four-dimensional (symplectic) representation is $\Lambda = (1, 0)$, for the five-dimensional (orthogonal) it is $\Lambda = (0, 1)$, and $\Lambda = (2, 0)$ for the ten-dimensional adjoint representation.

Now consider an irreducible finite-dimensional, analytic representation $\sigma_\Lambda : \mathrm{Sp}(4) \to \mathrm{GL}(V)$. Then V can be decomposed into a direct sum of the weight spaces relative to a (chosen and fixed) maximal torus of Sp(4) as

$$V = \bigoplus_{\lambda \in \Omega_\Lambda} V^\lambda. \tag{2.9}$$

The character of this representation σ_Λ is defined by

$$\chi_\Lambda : \mathrm{Sp}(4) \longrightarrow \mathbb{C}$$

$$x \longmapsto \chi_\Lambda(x) := \mathrm{Tr}(\sigma_\Lambda(x)). \tag{2.10}$$

If x is an element of the maximal torus of Sp(4), then it can be written as $x = \exp(2\pi \mathrm{i} \boldsymbol{x})$, $\boldsymbol{x} \in \mathfrak{sp}_4(\mathbb{C})$, so that

$$\chi_\Lambda(x) = \sum_{\lambda \in \Omega_\Lambda} \dim V^\lambda \, \mathrm{e}^{2\pi \mathrm{i}\langle \lambda, \boldsymbol{x}\rangle} \tag{2.11}$$

where $\langle \cdot, \cdot \rangle : \mathfrak{sp}_4(\mathbb{C})^* \times \mathfrak{sp}_4(\mathbb{C}) \to \mathbb{C}$ is the canonical pairing (of the vector space $\mathfrak{sp}_4(\mathbb{C})$ and its dual). For our purposes, x is an element of finite order (EFO) [9, 10, section 4]. The link between the value of a character on an EFO and the signature of a finite representation of SU(n) has been considered in [10, section 9.2]. Results similar to ours can be derived for other non-compact Lie groups.

3. Characters values on elements of finite order

As mentioned previously, we first consider the signatures as obtained from the character values on specific EFOs. Here, we must look for the two EFO of the simply connected compact simple Lie group Sp(4) which generate the signatures of the real forms of SO(4, 1) and SO(3, 2). Because the characters are invariant under the adjoint action of Sp(4), we shall be interested in *conjugacy classes* of EFOs. These classes are in one-to-one correspondence with the *Kac's coordinates* $\boldsymbol{s} = [s_0, s_1, \ldots, s_l]$ (l being the rank of the underlying group K) of non-negative integers with 1 being the greatest common divisor [9, 10, section 4]. These numbers are attached to the nodes of the extended Coxeter–Dynkin diagram. The EFO has within its conjugacy class a unique diagonal representative which acts on a weight space $V^\lambda(\lambda = \sum_{i=1}^{l} c_i\alpha_i \in \Omega_\Lambda, \alpha_i$ being the positive simple roots, $c_i \in \mathbb{Q})$ of any representation Λ of K as)

$$v \to \exp\left(\frac{2\pi \mathrm{i}}{M}\langle \lambda, \boldsymbol{s}\rangle\right) v \qquad v \in V^\lambda. \tag{3.1}$$

In the case of $\mathfrak{sp}_4(\mathbb{C})$, for which $K = \mathrm{Sp}(4)$, one has $l = 2$ and

$$M = s_0 + 2s_1 + s_2. \tag{3.2}$$

The expression $\langle \lambda, \boldsymbol{s}\rangle$ (where $\lambda = c_1\alpha_1 + c_2\alpha_2$) is evaluated through $\langle \alpha_i, \boldsymbol{s}\rangle = s_i$, so that

$$\langle \lambda, \boldsymbol{s}\rangle = c_1 s_1 + c_2 s_2. \tag{3.3}$$

Hereafter, we shall denote the value of the *character* χ_Λ on the element $\boldsymbol{s} = [s_0, s_1, s_2]$ in the representation $\Lambda = (\lambda_1, \lambda_2)$ by $\chi_{(\lambda_1,\lambda_2)}([s_0, s_1, s_2])$.

As mentioned in [12], the characters of some EFOs can be interpreted as generators of the signatures of representations of the non-compact real forms. It is easy to find which EFO generates the signatures of a given real form, by looking at the defining five-dimensional representation $\sigma_{(0,1)}$. For this representation, the diagonal representative of the conjugacy class of the EFO denoted $\boldsymbol{s} = [s_0, s_1, s_2]$ is

$$\mathrm{diag}(\mu^{s_1+s_2}, \mu^{s_1}, \mu^0, \mu^{-s_1}, \mu^{-(s_1+s_2)}) \tag{3.4}$$

with $\mu = \exp(2\pi \mathrm{i}/M)$. From (3.2), we see that the two classes of EFOs which provide us with the signatures are $\boldsymbol{s} = [0, 1, 0]$ and $[1, 0, 1]$. Upon substitution into (3.4) we get, for $\boldsymbol{s} = [0, 1, 0]$,

$$\mathrm{diag}(-1, -1, 1, -1, -1) \tag{3.5}$$

which corresponds to SO(4, 1), and for $\boldsymbol{s} = [1, 0, 1]$,

$$\mathrm{diag}(-1, 1, 1, 1, -1) \tag{3.6}$$

which leads to SO(3, 2). We note that there is a different sign between the diagonal representative of the EFO, and the corresponding bilinear form. This sign is immaterial for our needs as the resulting groups SO(2, 3) (respectively SO(1, 4)) are isomorphic to SO(3, 2) (respectively SO(4, 1)) and, therefore, so is the sign of the signature itself.

To summarize, the correspondence between the EFOs and the non-compact real form whose signature the EFO's character generator is associated with is

$$\begin{aligned} \boldsymbol{s} &= [0, 1, 0] \leftrightarrow \mathrm{SO}(4, 1) \\ \boldsymbol{s} &= [1, 0, 1] \leftrightarrow \mathrm{SO}(3, 2). \end{aligned} \tag{3.7}$$

Up to a factor of -1, the character value of the representation $\Lambda = (\lambda_1, \lambda_2)$ on an EFO $\boldsymbol{s}$ is given by

$$\chi_{(\lambda_1,\lambda_2)}(\boldsymbol{s}) = \sum_{\lambda\in\Omega_\Lambda} \exp\left(\frac{2\pi\mathrm{i}\langle\lambda, \boldsymbol{s}\rangle}{M}\right) \tag{3.8}$$

with $\boldsymbol{s}$ as in (3.7). For instance, for the four-dimensional representation $\Lambda = (1, 0)$, we get, using (3.8)

$$\begin{aligned} \chi_{(1,0)}(\boldsymbol{s}) = \exp\left[\frac{2\pi\mathrm{i}}{M}\left(s_1 + \frac{s_2}{2}\right)\right] + \exp\left[\frac{2\pi\mathrm{i}}{M}\left(\frac{s_2}{2}\right)\right] + \exp\left[-\frac{2\pi\mathrm{i}}{M}\left(\frac{s_2}{2}\right)\right] \\ + \exp\left[-\frac{2\pi i}{M}\left(s_1 + \frac{s_2}{2}\right)\right] = 2\cos\left[\frac{2\pi}{M}\left(s_1 + \frac{s_2}{2}\right)\right] + 2\cos\left[\frac{2\pi}{M}\left(\frac{s_2}{2}\right)\right] \end{aligned} \tag{3.9}$$

so that

$$\chi_{(1,0)}([0, 1, 0]) = 0 = \chi_{(1,0)}([1, 0, 1]).$$

Thus, the signature of the four-dimensional symplectic representation is zero for both de Sitter groups.

For the five-dimensional representation $\Lambda = (0, 1)$, we have

$$\begin{aligned} \chi_{(0,1)}(\boldsymbol{s}) = \exp\left[\frac{2\pi\mathrm{i}}{M}(s_1 + s_2)\right] + \exp\left[\frac{2\pi\mathrm{i}}{M}(s_1)\right] + \exp[0] + \exp\left[-\frac{2\pi\mathrm{i}}{M}(s_1)\right] \\ + \exp\left[-\frac{2\pi\mathrm{i}}{M}(s_1 + s_2)\right] = 1 + 2\cos\left[\frac{2\pi}{M}(s_1 + s_2)\right] + 2\cos\left[\frac{2\pi}{M}(s_1)\right] \end{aligned} \tag{3.10}$$

so that

$$\chi_{(0,1)}([0, 1, 0]) = -3 \qquad \chi_{(0,1)}([1, 0, 1]) = 1.$$

For the adjoint representation $\Lambda = (2, 0)$, the equation (3.8) gives

$$\chi_{(2,0)}(\boldsymbol{s}) \quad = \quad 2\left\{\cos\left[\frac{2\pi}{M}(2s_1+s_2)\right]+\cos\left[\frac{2\pi}{M}(s_1+s_2)\right]+\cos\left[\frac{2\pi}{M}s_1\right]+\cos\left[\frac{2\pi}{M}s_2\right]+1\right\} \tag{3.11}$$

so that

$$\chi_{(2,0)}([0, 1, 0]) = 2 = -\chi_{(2,0)}([1, 0, 1]).$$

However, this approach is very tedious when it comes to higher-dimensional representations. Then, it is useful to decompose the representation space into a sum over Weyl group orbits as

$$V^\Lambda = \bigoplus_{\lambda\in\Omega_\Lambda} V^\lambda = \bigoplus_{j=1}^{k} V^{W\mu_j} \qquad \text{where } V^{W\mu_j} := \bigoplus_{\nu\in W\mu_j} V^\nu \tag{3.12}$$

where $W\mu_j$ is an orbit in Ω_Λ, that is, a set of weights such that for any $\lambda, \lambda' \in W\lambda_j$, there exists an element w of the Weyl group W of $\mathfrak{sp}_4(\mathbb{C})$, such that $\lambda' = w(\lambda)$. Each orbit $W\mu_j$ is labelled by its (unique) dominant weight μ_j. The k in (3.12) is the number of Weyl group orbits in the representation or the number of dominant weights in its weight system. Our construction is based on the fact that *all* V^μ, $\mu \in W\mu_j$ *have the same dimension* $m_\Lambda^{\mu_j}$ and thus (3.8) can be cast into the form

$$s_{\Omega_\Lambda} = \chi_\Lambda([s_0, s_1, s_2]) = \sum_{j=1}^{k} m_\Lambda^{\mu_j} s_{W\mu_j} \tag{3.13}$$

where

$$s_{W\mu_j} = \sum_{\mu \in W\mu_j} \exp\left(\frac{2\pi \mathrm{i}}{M}\langle \mu, s\rangle\right) \tag{3.14}$$

is the signature of the orbit and $m_\Lambda^{\mu_j}$ is its multiplicity, (i.e. the dimension of V^{μ_j}). Multiplicities can be efficiently calculated [16] or found in tables, for example, in [14].

The Lie algebra $\mathfrak{sp}_4(\mathbb{C})$ admits four types of Weyl group orbits, represented by their dominant weights:

$$(0,0), (a,0), (0,b), (c,d), \qquad \text{where } a, b, c, d \in \mathbb{N} - \{0\}. \tag{3.15}$$

The contribution $s_{W\mu_j}$ of the orbit $W\mu_j$ to the character value is found as indicated by (3.14). Consider, for instance, the detailed calculation of $s_{W(a,0)}$. Using (3.8), we find

$$\begin{aligned} s_{W(a,0)} &= \exp\left[\frac{2\pi \mathrm{i}a}{M}\left(s_1 + \frac{s_2}{2}\right)\right] + \exp\left[\frac{2\pi \mathrm{i}a}{M}\left(\frac{s_2}{2}\right)\right] + \exp\left[-\frac{2\pi \mathrm{i}a}{M}\left(\frac{s_2}{2}\right)\right] \\ &\quad + \exp\left[-\frac{2\pi \mathrm{i}a}{M}\left(s_1 + \frac{s_2}{2}\right)\right] = 2\cos\left[\frac{2\pi a}{M}\left(s_1 + \frac{s_2}{2}\right)\right] + 2\cos\left[\frac{2\pi a}{M}\left(\frac{s_2}{2}\right)\right] \end{aligned} \tag{3.16}$$

so that, for SO(4, 1), we get

$$\begin{aligned} s_{W(a,0)}([0,1,0]) &= 2\cos(\pi a) + 2 \\ &= \begin{cases} 4 & \text{for } a \text{ even} \\ 0 & \text{for } a \text{ odd} \end{cases} \\ &= 2[1 + (-1)^a]. \end{aligned} \tag{3.17}$$

For SO(3, 2), we have

$$\begin{aligned} s_{W(a,0)}([1,0,1]) &= 4\cos\left(\frac{\pi a}{2}\right) \\ &= \begin{cases} 4 & \text{for } a/2 \text{ even} \\ -4 & \text{for } a/2 \text{ odd} \\ 0 & \text{for } a \text{ odd} \end{cases} \\ &= 2[1 + (-1)^a](-1)^{a/2}. \end{aligned} \tag{3.18}$$

Proceeding similarly with the other orbits in (3.15), we have the characters displayed in table 1.

Another convenient way to compute the characters of representations of simple Lie groups is to use the generating functions called 'character generators'. Here again, this is from some EFOs' character generators that we can find the signatures. Depending on

Table 1. Character formulae for Weyl group orbits.

Orbit	$s = [0, 1, 0]$	$s = [1, 0, 1]$
$\langle 0, 0\rangle_s^W$	1	1
$\langle a, 0\rangle_s^W$	$2[1 + (-1)^a]$	$2[1 + (-1)^a](-1)^{a/2}$
$\langle 0, b\rangle_s^W$	$4(-1)^b$	$2[1 + (-1)^b]$
$\langle c, d\rangle_s^W$	$4[1 + (-1)^c](-1)^d$	$2[1 + (-1)^c][1 + (-1)^d](-1)^{c/2}$

the EFO, we will find the signatures of SO(4, 1), or the signatures of SO(3, 2). From the table V of [12], the character generators of interest are

$$\mathrm{SO}(4,1):\qquad \frac{(1 + A^2B)}{(1 - A^2)^2(1 + B)^3} \tag{3.19}$$

and

$$\mathrm{SO}(3,2):\qquad \frac{(1 + B)(1 + A^2B)}{(1 + A^2)^2(1 - B^2)^2}. \tag{3.20}$$

The signature of the representation $\Lambda = (\lambda_1, \lambda_2)$ is equal to the coefficient of the term $A^{\lambda_1}B^{\lambda_2}$ in the power expansion of the appropriate generating function (3.19) or (3.20). This procedure too can be very tedious in practice.

4. Signature formulae and examples

If we decompose the irreducible representation $V = V^\Lambda$ with highest weight Λ into its different kinds of Weyl group orbits

$$\begin{aligned} V^\Lambda &= \bigoplus_{j=1}^{k} V^{W\mu_j} \\ &= V^{(0,0)} \oplus \bigoplus_{\mu_j \text{ of form } (a,0)} V^{(a,0)} \oplus \bigoplus_{\mu_j \text{ of form } (0,b)} V^{(0,b)} \oplus \bigoplus_{\mu_j \text{ of form } (c,d)} V^{(c,d)} \end{aligned} \tag{4.1}$$

we find that the signature formula for SO(4, 1) is

$$\begin{aligned} s_\Lambda^{\mathrm{SO}(4,1)} = m_\Lambda^{(0,0)} &+ 2\sum_{(a,0)\in\{\mu_1,\ldots,\mu_k\}} \left[m_\Lambda^{(a,0)}[1 + (-1)^a]\right] + 4\sum_{(0,b)\in\{\mu_1,\ldots,\mu_k\}} \left[m_\Lambda^{(0,b)}(-1)^b\right] \\ &+4\sum_{(c,d)\in\{\mu_1,\ldots,\mu_k\}} \left[m_\Lambda^{(c,d)}[1 + (-1)^c](-1)^d\right] \end{aligned} \tag{4.2}$$

and for SO(3, 2), it is

$$\begin{aligned} s_\Lambda^{\mathrm{SO}(3,2)} = m_\Lambda^{(0,0)} &+ 2\sum_{(a,0)\in\{\mu_1,\ldots,\mu_k\}} \left[m_\Lambda^{(a,0)}[1 + (-1)^a](-1)^{a/2}\right] + 2\sum_{(0,b)\in\{\mu_1,\ldots,\mu_k\}} \left[m_\Lambda^{(0,b)}[1 + (-1)^b]\right] \\ &+2\sum_{(c,d)\in\{\mu_1,\ldots,\mu_k\}} \left[m_\Lambda^{(c,d)}[1 + (-1)^c]\ [1 + (-1)^d]\ (-1)^{c/2}\right]. \end{aligned} \tag{4.3}$$

The summation is taken over all the orbits. The signature formulae (4.2) and (4.3) are the central result of our paper.

We now turn to some specific examples by using the methods described in the previous section. First consider the case of the four-dimensional representation $\Lambda = (1, 0)$. We have seen that the EFO characters provide a zero signature for both de Sitter groups (see below

(3.8)). Let us recover this result by using the generating functions (3.19) and (3.20). To do so, we shall use repeatedly the power expansion

$$\begin{aligned}\frac{1}{1+x} &= \sum_{k=0}^{\infty}(-x)^k \\ &= 1 - x + x^2 - x^3 + x^4 - + \cdots \qquad |\, x \,| < 1.\end{aligned} \tag{4.4}$$

For the four-dimensional representation $\Lambda = (1, 0)$, we must look for the coefficient of the term A in (3.19) and (3.20). It is clear that for both generating functions, the A expansion begins a quadratic term, so that the coefficient of A (and therefore the signature of the representation) is zero, as obtained in (3.10).

For the five-dimensional representation $\Lambda = (0, 1)$, we have obtained the signature -3 for SO(4, 1), and 1 for SO(3, 2). We must look for the coefficient of B in the expansion of character generators. For SO(4, 1), we expand (3.19). The only chance to get a term in B is by expanding the $(1 + B)^3$ in the denominator. We have

$$\begin{aligned}(1+B)^{-3} &= (1 + 3B + 3B^2 + B^3)^{-1} \\ &\approx 1 - 3B - 3B^2 - B^3 - \cdots\end{aligned} \tag{4.5}$$

so that the signature is -3, as expected. For SO(3, 2), we use (3.20). We get a term in B by considering the $(1 + B)$ in the numerator, so that we get at once the value $+1$, which is the signature already obtained.

Our last example consists in the adjoint representation $\Lambda = (2, 0)$, for which we obtained the signature 2 for SO(4, 1), and -2 for SO(3, 2) (see below (3.11)). For SO(4, 1), we look for terms in A^2 in (3.19). The only possibility is through the term $(1 - A^2)^{-2}$, which gives

$$\begin{aligned}(1-A^2)^{-2} &= (1 - 2A^2 + A^4)^{-1} \\ &\approx 1 + 2A^2\end{aligned} \tag{4.6}$$

which provides 2 as the signature. For SO(3,2), we expand the term $(1 + A^2)^{-2}$ of (3.20), to obtain

$$\begin{aligned}(1+A^2)^{-2} &= (1 + 2A^2 + A^4)^{-1} \\ &\approx 1 - 2A^2\end{aligned} \tag{4.7}$$

so that we get -2.

We close this section by writing down explicitly the values of $p_\Lambda, q_\Lambda, s_\Lambda$ and d_Λ for direct sums and tensor products of a four-dimensional and a five-dimensional representation, by using (2.4)–(2.6). For SO(4, 1), we get for the direct sum $(1, 0) \oplus (0, 1)$

$$s_\Lambda = -3 \qquad d_\Lambda = 9 \qquad p_\Lambda = 3 \qquad q_\Lambda = 6 \tag{4.8}$$

and for the tensor product $(1, 0) \otimes (0, 1)$

$$s_\Lambda = 0 \qquad d_\Lambda = 20 \qquad p_\Lambda = 10 \qquad q_\Lambda = 10. \tag{4.9}$$

For SO(3, 2), the direct sum $(1, 0) \oplus (0, 1)$ has properties

$$s_\Lambda = 1 \qquad d_\Lambda = 9 \qquad p_\Lambda = 5 \qquad q_\Lambda = 4 \tag{4.10}$$

and for the tensor product $(1, 0) \otimes (0, 1)$, we get the same results as in (4.9).

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